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LETTER TO THE EDITOR

Exact soliton solutions for two spacetime dimensional boson–fermion system

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Abstract. The exact soliton solutions for the boson-fermion system are presented. They have the generalised kink solution character. A formal solution corresponding to the fractional charge is suggested.

We would like to present the soliton-like solutions for the classical non-linear scalar field φ coupled by the Yukawa interaction to the fermion field ψ. An interaction of this type is typical in physics and it appears, for example, in the Weinberg–Salam model where it describes the Higgs bosons coupled to the fermion field. The fascinating thing is that the soliton type bound states of this system can have a fractional effective charge (Jackiw and Rebbi 1976, Bullough and Caudrey 1980). The existence of such solutions in nature seems to be more fascinating still (Su *et al* 1979, 1980).

Let us consider the system with Lagrangian density as

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi+\mathrm{i}\bar{\psi}\not{\partial}\psi-\lambda(\varphi^2-a^2)^2-G\bar{\psi}\psi\varphi. \tag{1}$$

In the $(1+1)$-dimensional spacetime only the scalar field part of (1) with $G=0$ gives the so-called kink solution (Bullough and Caudrey 1980). The aim of this letter is to present explicitly the soliton solution for the full boson–fermion system (1).

In the $(1+1)$-dimensional case it is very useful to bosonise the fermion field (Bardeen *et al* 1983)

$$\mathrm{i}\bar{\psi}\not{\partial}\psi=\tfrac{1}{2}(\partial_\mu\sigma)^2 \tag{2a}$$

$$\bar{\psi}\psi=-C_\mu N^\mu\cos(2\sqrt{\pi}\sigma) \tag{2b}$$

$$\bar{\psi}\gamma_\mu\psi=(1/\sqrt{\pi})\varepsilon_{\mu\nu}\partial^\mu\sigma. \tag{2c}$$

This leads to the Lagrangian density of the two boson fields:

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\varphi)^2+\tfrac{1}{2}(\partial_\mu\sigma)^2-\lambda(\varphi^2-a^2)-2\sqrt{\pi}G\mu\cos(2\sqrt{\pi}\sigma)\varphi. \tag{3}$$

This Lagrangian density may be interpreted as the scalar and fermion field condensate Lagrangian density. This is the simplest model describing the fermion–soliton interaction which can lead to instability—the Callan–Rubakov effect (Bennett 1984). The electric charge

$$Q=\int_{-\infty}^{\infty}\mathrm{d}x\,\bar{\psi}\gamma_0\psi=\frac{1}{\sqrt{\pi}}\Delta\sigma=\frac{1}{\sqrt{\pi}}[\sigma(\infty)-\sigma(-\infty)] \tag{4}$$

is conserved in this system.

When the Yukawa interaction vanishes $(G=0)$ the chirality

$$\chi=\int_{-\infty}^{\infty} \mathrm{d}x\, \bar{\psi}\psi \qquad (5)$$

is also conserved.

The Euler–Lagrange equations for (3) are:

$$\Box\varphi+4\lambda(\varphi^2-a^2)\varphi+2\sqrt{\pi}G\mu\cos(2\sqrt{\pi}\sigma)\varphi=0 \qquad (6)$$

$$\Box\varphi-4\pi G\mu\sin(2\sqrt{\pi}\sigma)\varphi=0. \qquad (7)$$

The simplest solutions of these equations can be obtained in the case of $\sigma=$ constant if $\sin(2\sqrt{\pi}\sigma)=0$ in (7). This means that $\cos(2\sqrt{\pi}\sigma)=\pm1$ which gives us two solutions according to the chirality sign $(\chi=\pm\infty)$. Equation (6) now takes the form

$$\Box\varphi+4\lambda(\varphi^2-a^2)\varphi\pm G\mu 2\sqrt{\pi}=0. \qquad (8)$$

This equation can be regarded as the Euler–Lagrange equation for the effective Lagrangian describing only the scalar field

$$\mathscr{L}_{\mathrm{ef}}=\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-U^{\pm}_{\mathrm{ef}}(\varphi) \qquad (9a)$$

$$U^{\pm}_{\mathrm{ef}}(\varphi)=\lambda(\varphi^2-a^2)^2\pm G\mu 2\sqrt{\pi}\varphi \qquad (9b)$$

where the potential U_{ef} or V is asymmetric and depends on the fermion field chirality. The asymmetric Lagrangian and solitons were examined by Kuczyński and Mańka (1985). It ought to be emphasised that the change of the chirality sign means the scalar field reflection $(\varphi\to-\varphi)$. The potential $U^{\pm}_{\mathrm{ef}}$ is presented in figure 1. Let us make the field φ shift

$$\varphi=\tilde{\varphi}+b$$

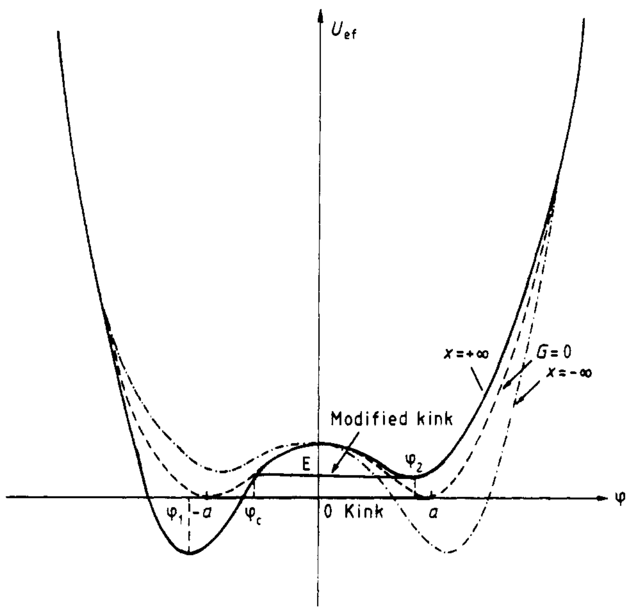

Figure 1. The asymmetric potential U_{ef}. The soliton corresponds to the segment E and the kink corresponds to the segment $(-a, +a)$ for $G=0$.

in order to remove the constant term in (8) which has the meaning of the coordinate system origin displacement to the extremum of U_{ef}. This demands b to fulfil the third-order equation

$$b^3 - a^2 b \pm G\mu\sqrt{\pi}/2\lambda = 0 \tag{10}$$

possessing two solutions if $G^2 = 16\lambda^2 a^6/\pi\mu^2$, three in the case $G^2 < 16\lambda^2 a^6/\pi\mu^2$ and one in the opposite case. After this shift equation (8) takes the following form:

$$\Box\tilde{\varphi} + 4\lambda\tilde{\varphi}^3 + 12\lambda b\tilde{\varphi}^2 + 4\lambda(3b^2 - a^2)\tilde{\varphi} = 0. \tag{11}$$

This equation is easy to integrate in the $(1+1)$-dimensional spacetime case. We obtain:

$$\int \frac{\mathrm{d}\tilde{\varphi}}{[\lambda\tilde{\varphi}^4 + 4b\lambda\tilde{\varphi}^3 + 2\lambda(3b^2 - a^2)\varphi^2 + A]^{1/2}} = \sqrt{2}\chi + C \tag{12}$$

where A is an arbitrary constant. Only in the case of the shift to the shallower minimum is this solution bounded and in the case when $A = 0$ this solution is not an oscillating type. The integral in (12) can be rewritten in the following form:

$$\int \frac{\mathrm{d}\tilde{\varphi}}{\tilde{\varphi}[\tilde{\varphi}^2 + 4b\tilde{\varphi} + 2(3b^2 - a^2)]^{1/2}} = \int \frac{\mathrm{d}\tilde{\varphi}}{\tilde{\varphi}[(\tilde{\varphi} - z_1)(\tilde{\varphi} - z_2)]^{1/2}}.$$

It is obvious that

$$z_1 = -2b + [2(a^2 - b^2)]^{1/2} \qquad z_2 = -2b - [2(a^2 - b^2)]^{1/2}$$

$$z_1 + z_2 = -4b \qquad z_1 z_2 = 2(3b^2 - a^2).$$

When $b \to a$ z_1 and z_2 tend to $-2a$. Using the first Euler substitution

$$[(\tilde{\varphi} - z_1)(\tilde{\varphi} - z_2)]^{1/2} = t - \tilde{\varphi} \tag{13}$$

we obtain the solution

$$\tilde{\varphi}_A = z_1 z_2 \frac{1 - \tanh^2(\psi/2)}{z_1 + z_2 - 2\sqrt{z_1 z_2}\tanh(\psi/2)} \tag{14}$$

with

$$\psi = (2\lambda z_1 z_2)^{1/2}(x - x_0).$$

This solution has an inflection point at $\psi = 0$. When z_1 tends to z_2 then $\tilde{\varphi} \to -a + a\tanh(\psi/2)$ which is the shifted kink. This shift is caused by translation of the origin of the coordinate system to the minimum. The second Euler substitution ($z_1 \neq z_2$)

$$[(\tilde{\varphi} - z_1)(\tilde{\varphi} - z_2)]^{1/2} = t(\tilde{\varphi} - z_1) \tag{15}$$

leads to the symmetric solution

$$\tilde{\varphi}_s = z_1 z_2 \frac{1 - \tanh^2(\psi/2)}{z_2 - z_1\tanh^2(\psi/2)} \tag{16}$$

with an extremum at $\psi = 0$. When $z_1 \to z_2$, $\tilde{\varphi}_s$ tends to the constant solution $\tilde{\varphi}_s = -2a$. The solution φ_s could be obtained from the solution $\tilde{\varphi}_A$ by shifting the coordinate centre from the inflection point $\psi = 0$ to the extremum:

$$\psi_0/2 = -\tanh^{-1}(\sqrt{z_1/z_2}).$$

The relation

$$\tilde{\varphi}_s(\psi/2) = \tilde{\varphi}_A(\tfrac{1}{2}\psi - \tfrac{1}{2}\psi_0) \tag{17}$$

is fulfilled. The extremum of the solution $\tilde{\varphi}_A$ depends on G and when $G \to 0$ or $z_1 \to z_2$, it tends to infinity whereas for the inflection point it is shifted. This coordinate system causes movement that tends to the constant equal to $-2a$, the solution for $G=0$. We notice that according to the chirality sign two solutions $\tilde{\varphi}_\pm$ exist. As $\Delta\sigma = 0$ both solutions have the electric charge $Q=0$. These solutions generalise the kink solution and when $G \to 0$ continuously tend towards the kink solution, but their shapes definitely differ from kink. They are one-dimensional equivalents of the dot or the bubble when $\chi = -\infty$ and $+\infty$, respectively. The deformed kink can be regarded as the bounded coherent state of scalar bosons and fermions, particles and antiparticles with total electric charge

$$Q = 0.$$

As both solutions $\tilde{\varphi}_+$ and $\tilde{\varphi}_-$ are symmetric with regard to the φ axis (see figure 2), we can obtain the new solution by piecing together different chirality solutions at the point x_0. The point x_0 is the solution of the equation

$$\tilde{\varphi}_+(+x_c) = 0.$$

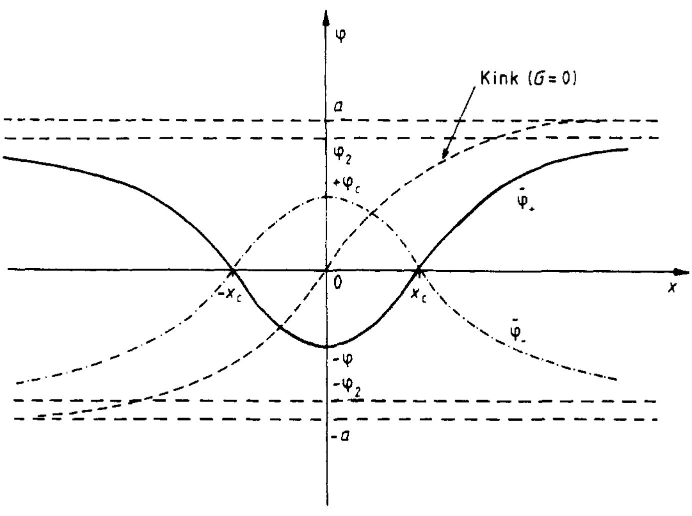

Figure 2. The soliton solution of the model presented for $x = -\infty$ and $+\infty$. The constant x_0 is chosen so that the minimum is at 0.

The joined solution has the form (see figure 3):

$$\tilde{\varphi}(x) = \begin{cases} \tilde{\varphi}_+(x) & x \geqslant x_0 \\ \tilde{\varphi}_-(x + 2x_0) & x < x_0. \end{cases} \tag{18}$$

However, for the fermion field there is a discontinuity. Indeed, for $\tilde{\varphi}_+(x)$, $\cos(2\sqrt{\pi}\sigma) = 1$ and for $\tilde{\varphi}_-(x)$, $\cos(2\sqrt{\pi}\sigma) = -1$ which means that

$$\sigma(x) = \begin{cases} 0 + 2\sqrt{\pi}k & x \geqslant x_0 \\ \tfrac{1}{2}\sqrt{\pi} + 2\sqrt{\pi}k & x < x_0 \end{cases} \tag{19}$$

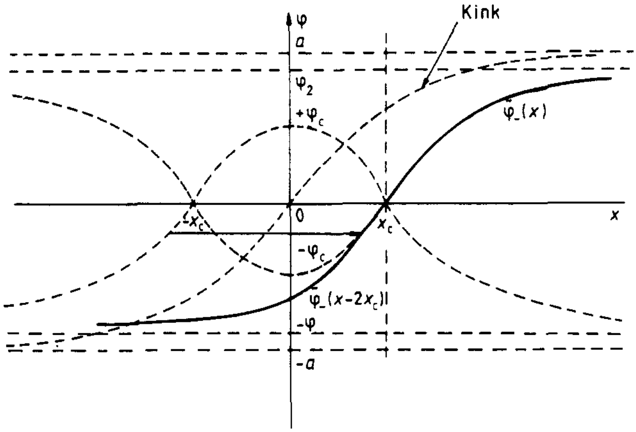

Figure 3. The fractional charge solution.

and $\Delta\sigma = -\frac{1}{2}\sqrt{\pi}$ which on the basis of (4) gives a fractional charge corresponding to this solution. By piecing together solutions at $-x$ we could obtain the $Q = \frac{1}{2}$ solution, so a fermion-boson soliton bound state with fractional total charge could be created. Even if this state could be unphysical this seems to be a very interesting and unusual fact and will be investigated in future papers.

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